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Inequivalence of the Massive Vector Meson and Higgs Models on a Manifold with Boundary

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Abstract

The exact quantization of two models, the massive vector meson model and the Higgs model in the London limit, both describing massive photons, is presented. Even though naive arguments (based on gauge-fixing) may indicate the equivalence of these models, it is shown here that this is not true in general when we consider these theories on manifolds with boundaries. We show, in particular, that they are equivalent only for a special choice of the boundary conditions that we are allowed to impose on the fields.

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1 Introduction

It is known that gauge theories involve only massless gauge bosons unless either the gauge symmetry is spontaneously broken [1] or topological terms [2, 3] are added to the action. Such models are not merely of theoretical interest. They have important applications in particle physics and in condensed matter physics. In the former they arise, for example, in models that incorporate the electroweak interactions [1]. Applications in condensed matter physics arise in situations such as superconductivity [4] where it is known that photons acquire a mass in the superconducting regime. Such models also arise in effective theories describing the long wavelength physics [5] of 2+1 dimensional systems.

It is usually believed that, under certain approximations, such theories involving massive gauge bosons are equivalent to the massive vector meson model. By the massive vector meson model is here meant the action whose gauge symmetry has been *explicitly* broken by the addition of a quadratic mass term [6]. The arguments that are used [1] to show the equivalence of these models depend quite crucially on gauge-fixing in some form or the other. On the other hand, we know that gauge-fixing arguments for manifolds with boundaries are always suspect because of the following two reasons.

Firstly, the Gauss law for manifolds with boundaries has to be defined by smearing it with appropriate test functions so as to ensure that they generate canonical transformations [7] on the phase space. Such a requirement restricts the allowed gauge transformations [7] which, by definition, are the canonical transformations generated by Gauss law. Since gauge-fixing arguments do not usually pay attention to this feature, they are not to be believed without further justifications. The second reason is a subtle one and is related to the fact that on a manifold with boundary, the Hamiltonian is self-adjoint and bounded from below only if the fields and their momenta satisfy suitable boundary conditions (BC's). Most of the gauge-fixing arguments either pick a special choice of BC's

or do not talk about it at all.

Because of the above reasons, it is clear that the equivalence of the two models is proven only on manifolds without boundaries (like $\mathbf{R}^n$). For manifolds with boundaries, a more careful analysis is warranted to display the similarities/differences between the massive vector meson model and a gauge theory with massive gauge bosons.

In Section 2, we look at the exact treatment (following Dirac) of the massive vector meson model. It is shown that the quantization here depends on a one-parameter family of BC's. We specialize to a particular BC for which we are able to carry out the quantization completely. We are not able to do this exact quantization for the most general BC's. In Section 3, we consider the Higgs model in the London limit (the modulus of the Higgs field is frozen to its vacuum expectation value). Here the exact treatment leads to a quadratic Hamiltonian along with a Gauss law. In this case, quantization depends on a two-parameter family of BC's and unlike the earlier case, we are able to carry through the exact quantization for the most general BC's. In Section 4, we compare the quantizations carried out in Sections 2 and 3. It turns out that there is a natural identification of the BC's of sections 2 and 3 provided one of the two parameters of section 3 is set to zero. We show that the Hamiltonian of the massive vector meson and Higgs models are different in general. In this Section, we also briefly compare these two theories with one other model describing a massive photon, namely the Maxwell-Chern-Simons (MCS) [2, 3] theory.

2 The Massive Vector Meson Model

In this section we will consider the usual Maxwell action augmented by a mass term for the vector potential A_μ on a space time manifold $D \times \mathbf{R}^1$, the two dimensional disc D

representing the spatial manifold and $\mathbf{R}^1$ denoting time:

$$S = \int_{D\times\mathbf{R}^1} d^3x \left\{ -\frac{1}{4e^2} F_{\mu\nu}F^{\mu\nu} - \frac{1}{2} m_A^2 A_\mu A^\mu \right\} \ . \tag{2.1}$$

Here $F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu$ is the electromagnetic tensor whose components are the electric field $E_i = \partial_0 A_i - \partial_i A_0$ and the magnetic field $B = \partial_1 A_2 - \partial_2 A_1 = \frac{1}{2}\epsilon_{ij}F_{ij}$ *.

It is well known [6] that this model describes a constrained system, with the second class constraints given by

$$\Pi_0 \approx 0 \quad \text{and} \quad m_A^2 A_0 + \partial_i \Pi_i \approx 0 \ , \tag{2.2}$$

$\Pi_\mu = (\Pi_0, \Pi_i)$ being the momenta conjugate to A_μ. Π_i here is related to E_i by $\Pi_i = \frac{1}{e^2}E_i$. Of course, the above constraints (2.2) have to be smeared with appropriate test functions so that they generate well-defined canonical transformations [7]. However, since these constraints are second class, they can be imposed strongly. Following Dirac's procedure [8] for the system described by (2.1), the Hamiltonian (up to irrelevant surface terms that depend on the test functions used to smear the above constraints) we end up with is

$$H = \frac{1}{2}\int_D d^2x \left\{ e^2\Pi_i^2 + \frac{1}{m_A^2}(\partial_i\Pi_i)^2 + \frac{1}{e^2}B^2 + m_A^2 A_i^2 \right\} \ , \tag{2.3}$$

where the variables A_i and Π_i satisfy the usual canonical commutation relations. For the following discussion it is convenient to rewrite the Hamiltonian (2.3) using the notation of differential forms †. After integrating by parts the second and third terms in (2.3) and neglecting the surface integrals, we get:

$$H = -\frac{1}{2}\int_D \left\{ \Pi * (e^2 + \frac{1}{m_A^2} d * d*)\Pi - A * (m_A^2 + \frac{1}{e^2} * d * d)A \right\} \ . \tag{2.4}$$

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*Throughout the paper we will use the three-dimensional metric η with $\eta_{00} = -1$, $\eta_{11} = \eta_{22} = +1$ and the three dimensional Levi-Civita symbol $\epsilon_{\lambda\mu\nu}$ with $\epsilon_{012} = +1$.

†We write the fields (A_1, A_2) , (Π_1, Π_2) as the one-forms $A = A_1 dx^1 + A_2 dx^2$, $\Pi = \Pi_1 dx^1 + \Pi_2 dx^2$ respectively and the fields B , $\partial_i \Pi_i$ as $*dA (= \frac{1}{2}\epsilon^{ij}F_{ij}$, $- * d * \Pi$ respectively. In the latter expression $*$ denotes the Hodge operation [9].

We can rewrite this expression in a very compact form if on the vector space of forms $\alpha^{(p)}$ of degree p we introduce the scalar product $<\alpha^{(p)},\beta^{(p)}>:=(-1)^p\int\overline{\alpha^{(p)}}*\beta^{(p)}$, where the bar denotes complex conjugation. Now (2.4) becomes

$$H=\frac{1}{2}<\Pi,(e^2+\frac{1}{m_A^2}d*d*)\Pi>+\frac{1}{2}<A,(m_A^2+\frac{1}{e^2}*d*d)A>\ . \tag{2.5}$$

In order to quantize this Hamiltonian, we need to expand the fields A and Π in a complete basis of the Hilbert space of one-forms. Since H is constructed from the differential operators $d*d*$ and $*d*d$, we would like to expand the fields in a basis of eigenfunctions of such operators [3]. In other words, we need to find a domain of self-adjointness[‡] for these operators. Let us first find a domain of self-adjointness [10] for the operator $*d*d$. From the relation:

$$0=<\alpha,*d*d\beta>-<*d*d\alpha,\beta>=\int_{\partial D}\left\{\overline{*d\alpha}\,\beta-\overline{\alpha}\,*d\beta\right\}\ , \tag{2.6}$$

where α,β are any two one-forms, it is easy to see that the operator $*d*d$ is self-adjoint on the domain

$$\mathcal{D}_\lambda=\{\alpha\ :\ *d\alpha\,|_{r=R}=-\lambda\alpha_\theta\,|_{r=R}\}\qquad\lambda\in\mathbf{R}^1\ . \tag{2.7}$$

To go from (2.6) to (2.7) we have required that the fields satisfy local rotationally invariant BC's. [By local BC's, we mean BC's which mix fields and their derivatives only at the same point.] In (2.7), r and θ are polar coordinates on the disc D with $r=R$ giving its boundary and $A_\theta=A_i\frac{\partial x^i}{\partial\theta}(r,\theta)$.

λ here can be any real number. But the Hamiltonian (2.5) is bounded from below only if $\lambda\geq 0$. This can be seen by noticing that

$$<\alpha,*d*d\alpha>=<d\alpha,d\alpha>-\int_{\partial D}\overline{\alpha}(*d\alpha)=<d\alpha,d\alpha>+\lambda\int_{\partial D}|\alpha_\theta|^2Rd\theta\ . \tag{2.8}$$

[‡]Let us recall that the property defining the domain $\mathcal{D}\subset\mathcal{H}$ of a self-adjoint operator T on a Hilbert space $\mathcal{H}$ is the following [10]: $<\chi,T\eta>-<T\chi,\eta>=0$, $\forall\eta\in\mathcal{D}\Leftrightarrow\chi\in\mathcal{D}$.

Since $< d\alpha, d\alpha > \geq 0$, $< \alpha, *d*d\alpha >$ is nonnegative iff $\lambda \geq 0$. Therefore, from now on, we will only consider the domains $\mathcal{D}_\lambda$ with $\lambda \geq 0$. Let us now turn to the problem of solving the eigenvalue equation

$$* d * dA = \omega^2 A \tag{2.9}$$

for the one-form A satisfying the BC's

$$* \, dA \,|_{r=R} = -\lambda A_\theta \,|_{r=R} \quad , \quad \lambda \geq 0 \; . \tag{2.10}$$

This problem has already been examined and solved in [3]. Here, we will not repeat the calculations and will only list the eigenmodes of $*d*d$, together with the corresponding eigenvalues ω^2.

The solutions for $\omega^2 \neq 0$ are of the form:

$$\left.\begin{array}{l} \Psi^{(1)}_{nm} = \mathcal{N}^{(1)}_{nm} * d[e^{in\theta} J_n(\omega^{(1)}_{nm} r)] \\ \Psi^{(1)}_{-nm} = \overline{\Psi^{(1)}_{nm}} \end{array}\right\} n \geq 0 \, , \, m > 0 \; , \tag{2.11}$$

where $J_n(x)$ is the real Bessel function of order n[†] and the eigenvalues $\omega = \omega^{(1)}_{nm}$ are fixed by the BC's (2.10) which now read:

$$\omega^{(1)}_{nm} J_n(\omega^{(1)}_{nm} R) = \lambda \left[\frac{d}{d(\omega^{(1)}_{nm} r)} J_n(\omega^{(1)}_{nm} r) \right]_{r=R} \; . \tag{2.12}$$

There is also a set of zero modes, solutions of (2.9) with $\omega = 0$, given by:

$$\left.\begin{array}{l} \Psi^{(0)}_{nm} = \mathcal{N}^{(0)}_{nm} \, d[e^{in\theta} J_n(\omega^{(0)}_{nm} r)] \\ \Psi^{(0)}_{-nm} = \overline{\Psi^{(0)}_{nm}} \end{array}\right\} n \geq 0 \, , \, m > 0 \; , \tag{2.13}$$

the Bessel functions now satisfying the condition $J_n(\omega^{(0)}_{nm} R) = 0$.

The functions (2.11) and (2.13) form a complete set of solutions for (2.9),(2.10) if $\lambda > 0$. On the contrary, if $\lambda = 0$ there is another set of zero modes, given by the so-called harmonic forms

$$h_n = \mathcal{N}^{(h)}_n \, dz^n \quad , \quad \overline{h_n} = \mathcal{N}^{(h)}_n \, d\overline{z}^n \qquad n > 0 \tag{2.14}$$

[†]Here and in the following, we adopt the convention that the normalization contants, such as $\mathcal{N}^{(1)}_{nm}$ in (2.11), are fixed by the conditions $< \Psi_{nm}, \Psi_{nm} >= 1$ and $\mathcal{N}^{(1)}_{nm} > 0$.

where $z = x_1 + ix_2 = re^{i\theta}$ is the complex coordinate on D.

It is important to notice that the harmonic functions are eigenfunctions of the operator $*$, $*h_n = ih_n$, and that, in addition, if $\lambda = 0$ there exists a relationship between the nonzero modes (2.11) and the zero modes (2.13) $\Psi^{(1)}_{nm} = *\Psi^{(0)}_{nm}$ with $\omega^{(1)}_{nm} = \omega^{(0)}_{nm}$. These two relations imply that, for $\lambda = 0$, the complete set of eigenfunctions $(\Psi^{(1)}_{nm}, \Psi^{0}_{nm}, h_n)$ of the operator $*d*d$ is also a complete set of eigenfunctions of the operator $d*d*$. Indeed, to diagonalise H, we can expand the fields A and Π in (2.5) as

$$\begin{aligned} A &= q^{(1)}_{nm}\Psi^{(1)}_{nm} + q^{(0)}_{nm}\Psi^{(0)}_{nm} + q^{(h)}_n h_n + c.c.\ , \\ \Pi &= p^{(1)}_{nm}\Psi^{(1)}_{nm} + p^{(0)}_{nm}\Psi^{(0)}_{nm} + p^{(h)}_n h_n + c.c.\ , \end{aligned} \tag{2.15}$$

where (as in the following) repeated indices are summed over. The Hamiltonian is then

$$\begin{aligned} H &= \frac{1}{2}\left\{\left[\left(e^2 + \frac{\omega^2_{nm}}{m^2_A}\right) p^{(0)\dagger}_{nm} p^{(0)}_{nm} + m^2_A q^{(0)\dagger}_{nm} q^{(0)}_{nm}\right] + \left[e^2 p^{(1)\dagger}_{nm} p^{(1)}_{nm} + \left(\frac{\omega^2_{nm}}{e^2} + m^2_A\right) q^{(1)\dagger}_{nm} q^{(1)}_{nm}\right] + \right. \\ &+ \left.\left[e^2 p^{(h)\dagger}_n p^{(h)}_n + m^2_A q^{(h)\dagger}_n q^{(h)}_n\right]\right\}\ , \end{aligned} \tag{2.16}$$

where the only non zero commutation relations satisfied by the operators $q^{(j)}$'s and $p^{(j)}$'s $(j = 0, 1, h)$ are $\left[q^{(1)}_{nm}, p^{(1)\dagger}_{n'm'}\right] = i\delta_{nn'}\delta_{mm} = \left[q^{(0)}_{nm}, p^{(0)}_{n'm'\dagger}\right]$, $\left[q^{(h)}_n, p^{(h)\dagger}_{n'}\right] = i\delta_{nn'}$.

Here $\omega_{nm} = \omega^{(0)}_{nm} = \omega^{(1)}_{nm}$ while the commutation relations follow from those of the variables A_i, Π_i. Let us define the annihilation-creation operators

$$\left.\begin{aligned} a^{(j)}_{nm} &= \tfrac{1}{\sqrt{2}}\left[\tfrac{1}{\sqrt{\Omega^{(j)}_{nm}}} p^{(j)}_{nm} - i\sqrt{\Omega^{(j)}_{nm}} q^{(j)}_{nm}\right] \\ a^{(j)\dagger}_{nm} &= \tfrac{1}{\sqrt{2}}\left[\tfrac{1}{\sqrt{\Omega^{(j)}_{nm}}} p^{(j)\dagger}_{nm} + i\sqrt{\Omega^{(j)}_{nm}} q^{(j)\dagger}_{nm}\right] \end{aligned}\right\} j = 0, 1 \tag{2.17}$$

$$\begin{aligned} a^{(h)}_n &= \tfrac{1}{\sqrt{2}}\left[\tfrac{1}{\sqrt{\Omega^{(h)}_n}} p^{(h)}_n - i\sqrt{\Omega^{(h)}_n} q^{(h)}_n\right] \\ a^{(h)\dagger}_n &= \tfrac{1}{\sqrt{2}}\left[\tfrac{1}{\sqrt{\Omega^{(h)}_n}} p^{(h)\dagger}_n + i\sqrt{\Omega^{(h)}_n} q^{(j)\dagger}_n\right] \end{aligned}\ , \tag{2.18}$$

with commutators $\left[a^{(j)}_{nm}, a^{(j)\dagger}_{n'm'}\right] = i\delta_{nn'}\delta_{mm'}$ $(j = 0, 1)$, $\left[a^{(h)}_n, a^{(h)\dagger}_{n'}\right] = i\delta_{nn'}$, where

$$\Omega^{(1)}_{nm} = \Omega^{(0)}_{nm} = \sqrt{\omega^{(0)2}_{nm} + e^2 m^2_A}\ ,\ \ \Omega^{(h)}_n = em_A\ . \tag{2.19}$$

Then (2.16) becomes

$$H = \Omega^{(1)}_{nm}\alpha^{(1)\dagger}_{nm}\alpha^{(1)}_{nm} + \Omega^{(0)}_{nm}\alpha^{(0)\dagger}_{nm}\alpha^{(0)}_{nm} + \Omega^{(h)}_{n}a^{\dagger}_{n}a_{n}\ . \tag{2.20}$$

The spectrum of H can be read off from (2.20).

We remark here that the lowest energy modes are the ones corresponding to the harmonic functions and are all degenerate, having an energy $\Omega^{(h)}_{n}$ that depends only on the mass m_A of the vector potential, for all n.

We conclude this section by looking briefly at the $\lambda > 0$ case. Now it is no longer true that the operator $d * d*$ is diagonal in the basis $(\Psi_{nm}, \Psi^{(0)}_{nm})$ of eigenfunctions of $*d * d$. If we continue to use a field expansion similar to (2.15), with the harmonic modes now missing, we can write (2.5) in the non-diagonal form

$$\begin{aligned} H \;=\; & \frac{1}{2}\left\{\left[\left(e^2\delta_{mm'} + \frac{\omega^{(0)2}_{nm'}f_{mm'}}{m_A^2}\right)p^{(0)\dagger}_{nm}p^{(0)}_{nm'} + m_A^2 q^{(0)\dagger}_{nm}q^{(0)}_{nm}\right] + \right. \\ & \left.\left[e^2 p^{(1)\dagger}_{nm}p^{(1)}_{nm} + \left(\frac{\omega^{(1)2}_{nm}}{e^2} + m_A^2\right)q^{(1)\dagger}_{nm}q^{(1)}_{nm}\right]\right\} \end{aligned} \tag{2.21}$$

where the overlap coefficients $f_{mm'} :=< \Psi^{(1)}_{nm}, *\Psi^{(1)}_{nm'} >$ are different from zero for every m, m'. Thus, each of the $p^{(0)}_{nm}$ mode of the momentum field is coupled to an infinite number of other such modes. We do not know how to diagonalize (2.21).

3 The Higgs Model

As before, we will work on the space-time manifold $D \times \mathbf{R}^1$. Consider then a $U(1)$ Higgs model with the modulus ρ of the Higgs field $\phi = \rho e^{iq\psi}$ frozen to its vacuum value. In this limit (the London limit), in addition to the vector potential A_μ, the only other degree of freedom is the real phase ψ. The action in these variables is then:

$$S = \int_{D\times\mathbf{R}^1} d^3x \left\{-\frac{1}{4e^2}F_{\mu\nu}F^{\mu\nu} - \frac{m_H^2}{2}(\partial_\mu\psi - A_\mu)(\partial^\mu\psi - A^\mu)\right\} \tag{3.1}$$

where $mHe = q\rho$ is the mass of the vector meson A_μ.

The Hamiltonian corresponding to (3.1) is

$$H = \int_D d^2x \left\{ \frac{e^2}{2}\Pi_i^2 + \frac{1}{2e^2}B^2 + \frac{1}{2m_H^2}\Pi^2 + \frac{m_H^2}{2}(\partial_i\psi - A_i)^2 \right\} \tag{3.2}$$

where $\Pi_i = \frac{E_i}{e^2}$ is the momentum conjugate to A_i and Π the momentum conjugate to ψ. This Hamiltonian has to supplemented by the Gauss law $\Pi - \partial_i\Pi_i \approx 0$. The non-zero Poisson Brackets are

$$\{\psi(x), \Pi(y)\} = \delta^2(x-y) \quad , \quad \{A_i(x), \Pi_j(y)\} = \delta_{ij}\delta^2(x-y) \; . \tag{3.3}$$

As explained in [7] and after equation (2.2), the correct way of reading the Gauss law is by smearing it with a test function $\Lambda^{(0)}$:

$$\mathcal{G}(\Lambda^{(0)}) = \int_D d^2x \Lambda^{(0)} \left(\Pi - \partial_i\Pi_i\right) = 0 \; , \tag{3.4}$$

where $\Lambda^{(0)}$ is zero on the boundary of the disc, $\Lambda^{(0)}|_{\partial D} = 0$.

Let us rewrite both (3.2) and (3.4) using the form notation. To do so, let us introduce the space of vectors (ψ, A) where ψ is a zero-form and A is a one-form, with the scalar product $<(\psi, A), (\psi', A')>=<\psi, \psi'>_0 + <A, A'>_1$ where $<\cdot,\cdot>_0$, $<\cdot,\cdot>_1$ are the scalar products respectively on zero and one forms, as previously defined. In addition, to simplify the notation, we set $f := m_H\psi$, $P := \Pi/m_H$, $\mathcal{A}_i := A_i/e$ and $\mathcal{E}_i := e\Pi_i$.

After integrating (3.2) by parts and neglecting surface terms, the Hamiltonian becomes

$$\begin{aligned} H &= \frac{1}{2}(<(f,\mathcal{A}), \hat{H}_0(f,\mathcal{A})> + <(P,\mathcal{P}),(P,\mathcal{E})>) \\ \hat{H}_0 &= \begin{bmatrix} *d*d & -m_He*d* \\ -m_Hed & *d*d + m_H^2e^2 \end{bmatrix} \end{aligned} \tag{3.5}$$

while the Gauss law (3.4) reads

$$\mathcal{G}(\Lambda^{(0)}) =< \Lambda^{(0)}, P + \frac{1}{m_He} * d * \mathcal{E} >= 0 \; . \tag{3.6}$$

Our analysis will now proceed as in the massive vector meson case. We have first to look for suitable BC's on the fields $(f, \mathcal{A})$ that make the Hamiltonian H diagonalizable. Let f, g be zero-forms and $\mathcal{A}, \mathcal{B}$ be one-forms. Then, from

$$
\begin{aligned}
& < (g,\mathcal{B}), \hat{H}_0(f,\mathcal{A}) > - < \hat{H}_0(g,\mathcal{B}),(f,\mathcal{A}) >= \\
& = \int_{\partial D} \left\{ \overline{*d\mathcal{B}}\,\mathcal{A} - \overline{\mathcal{B}} * d\mathcal{A} \right\} - \int_{\partial D} \left\{ \overline{g} * (df - m_H e\mathcal{A}) - \overline{*(dg - m_H e\mathcal{B})}\, f \right\}, \quad (3.7)
\end{aligned}
$$

we see that $\hat{H}_0$ is self-adjoint on the domain

$$
\mathcal{D}_{\lambda\mu} = \{(f,\mathcal{A}) \ : \ *d\mathcal{A}\,|_{r=R} = -\lambda \mathcal{A}_\theta\,|_{r=R} \text{ and } f\,|_{r=R} = -\mu[*(df - m_H e\mathcal{A})]_\theta\,|_{r=R}\} \quad \lambda,\mu \in \mathbf{R}^1. \tag{3.8}
$$

We note here that λ has the same role as the λ that appeared in section 2, while μ is a new parameter that did not exist in the previous case. We have as before imposed locality and rotational invariance to obtain (3.8).

On this domain, the Hamiltonian can be rewritten as

$$
\begin{aligned}
H &= \frac{1}{2}\{< \mathcal{E},\mathcal{E} >_1 + < P,P >_0 + < d\mathcal{A}, d\mathcal{A} >_1 + < df - m_H e\mathcal{A}, df - m_H e\mathcal{A} >\} + \\
&+ \frac{1}{2}\int_{\partial D} R d\theta \left\{ \lambda |\mathcal{A}_\theta|^2 + \frac{1}{\mu} f^2 \right\}_{r=R}, \qquad (3.9)
\end{aligned}
$$

so that it is nonnegative iff $\lambda, \mu \geq 0$. Thus from now on we will consider only domains $\mathcal{D}_{\lambda\mu}$ with $\lambda \geq 0$, $\mu \geq 0$.

Our task then is to solve the eigenvalue problem

$$
\hat{H}_0 \begin{pmatrix} f \\ \mathcal{A} \end{pmatrix} = \omega^2 \begin{pmatrix} f \\ \mathcal{A} \end{pmatrix} \tag{3.10}
$$

with the BC's

$$
\begin{aligned}
* \, d\mathcal{A}\,|_{r=R} &= -\lambda \mathcal{A}_\theta\,|_{r=R} && (3.11) \\
f\,|_{r=R} &= -\mu[*(df - m_H e\mathcal{A})]_\theta\,|_{r=R} \,. && (3.12)
\end{aligned}
$$

The eigenmodes of (3.10) subject to the above BC's can be found by an analysis very similar to that used to solve (2.9).

If $\omega^2 \neq 0$, this system of equations can be decoupled into one differential equation for $\mathcal{A}$ and one equation defining f as a function of $\mathcal{A}$:

$$(d * d * + * d * d)\mathcal{A} = (\omega^2 - m_H^2 e^2)\mathcal{A} \,, \tag{3.13}$$

$$f = -\frac{1}{m_H e} * d * \mathcal{A} \,. \tag{3.14}$$

Let us first look at the modes of $\mathcal{A}$ obtained from equation (3.13) when $\omega^2 = m_H^2 e^2$. In this case the harmonic one forms (2.14) satisfy (3.13) and the BC (3.11) if $\lambda = 0$. In addition, from (3.14) and (2.14) it follows that $f \equiv 0$, so that (3.12) is satisfied only for $\mu = 0$. This means that the Hamiltonian (3.5) admits harmonic modes for $\lambda = 0$ iff μ is also zero. Since we are interested in comparing the Higgs model in the broken phase with the massive vector meson model and since the latter does admit harmonic modes for $\lambda = 0$, from now on we will set $\mu \equiv 0$.

Thus, if $\lambda = 0$, we have a set of solutions $(f, \mathcal{A})$ of (3.10) corresponding to the eigenvalue $\omega^2 = m_H^2 e^2$ given by

$$H_n = \mathcal{M}_n^{(h)} \ (0, dz^n) \quad , \quad \overline{H_n} = \mathcal{M}_n^{(h)} \ (0, d\overline{z}^n) \qquad n > 0 \,. \tag{3.15}$$

If $\omega^2 > m_H^2 e^2$, (3.13) and (3.14) admit the following two sets of solutions for $\lambda \geq 0$:

$$\left. \begin{array}{l} \Phi_{nm}^{(\alpha)} = \mathcal{M}_{nm}^{(\alpha)} \left(-\frac{1}{m_H e} e^{in\theta} J_n(\alpha_{nm} r), \frac{1}{\alpha_{nm}^2} d[e^{in\theta} J_n(\alpha_{nm} r)] \right) \\ \Phi_{-nm}^{(\alpha)} = \overline{\Phi_{nm}^{(\alpha)}} \end{array} \right\} n \geq 0 \,,\, m > 0 \,, \tag{3.16}$$

$$\left. \begin{array}{l} \Phi_{nm}^{(\beta)} = \mathcal{M}_{nm}^{(\beta)} \left(0, \frac{1}{\beta_{nm}^2} * d[e^{in\theta} J_n(\beta_{nm} r)] \right) \\ \Phi_{-nm}^{(\beta)} = \overline{\Phi_{nm}^{(\beta)}} \end{array} \right\} n \geq 0 \,,\, m > 0 \,. \tag{3.17}$$

corresponding to the eigenvalues $\omega_{nm}^{(\alpha)2} = \alpha_{nm}^2 + m_H^2 e^2$ and $\omega_{nm}^{(\beta)2} = \beta_{nm}^2 + m_H^2 e^2$ respectively, where α_{nm} and β_{nm} are determined by the BC's (3.11) and (3.12) with $\mu = 0$:

$$J_n(\alpha_{nm} R) = 0 \,, \tag{3.18}$$

$$\beta_{nm} J_n(\beta_{nm} R) \;=\; \lambda \left[\frac{d}{d(\beta_{nm} r)} J_n(\beta_{nm} r) \right]_{r=R} . \qquad (3.19)$$

(Notice that the α_{nm} and β_{nm} above are respectively identical to the $\omega^{(0)}_{nm}$ and $\omega^{(1)}_{nm}$ of Section 2.)

Finally we obtain a set of solutions to (3.10) when $\omega^2 = 0$, which are given by:

$$\left.\begin{array}{l} \Phi^{(0)}_{nm} = \mathcal{M}^{(0)}_{nm} \left(e^{in\theta} J_n(\gamma_{nm} r), \frac{1}{m_H e} d[e^{in\theta} J_n(\gamma_{nm} r)] \right) \\ \Phi^{(0)}_{-nm} = \overline{\Phi^{(0)}_{nm}} \end{array} \right\} n \geq 0 \, , \, m > 0 \, , \qquad (3.20)$$

with the γ_{nm}'s fixed by the condition $J_n(\gamma_{nm} R) = 0$ (so that $\gamma_{nm} = \alpha_{nm}$).

In conclusion, for $\lambda = 0$, (3.15,3.16,3.17,3.20) form a complete set of eigenfunctions that allow us to expand the fields $(f, \mathcal{A})$ and $(P, \mathcal{E})$ as

$$\begin{aligned} (f, \mathcal{A}) &= q^{(\alpha)}_{nm} \Phi^{(\alpha)}_{nm} + q^{(\beta)}_{nm} \Phi^{(\beta)}_{nm} + q^{(0)}_{nm} \Phi^{(0)}_{nm} + q^{(h)}_{n} H_n + c.c. \\ (P, \mathcal{E}) &= p^{(\alpha)}_{nm} \Phi^{(\alpha)}_{nm} + p^{(\beta)}_{nm} \Phi^{(\beta)}_{nm} + p^{(0)}_{nm} \Phi^{(0)}_{nm} + p^{(h)}_{n} H_n + c.c. \end{aligned} \qquad (3.21)$$

where, by virtue of (3.3), the only nonzero commutation relations are $\left[q^{(j)}_{nm}, p^{(j)}_{n'm'} \right] = i\delta_{nn'}\delta_{mm'}$, $\left[q^{(h)}_{n}, p^{(h)}_{n'} \right] = i\delta_{nn'}$, where $j = \alpha, \beta$ or 0.

If $\lambda > 0$, the field expansions look very similar to (3.21), except for the fact that in this case the harmonic modes H_n are absent. We now will turn our attention to the Gauss law (3.6). Since in (3.6) the test function $\Lambda^{(0)}$ vanishes on the boundary ∂D, it can be chosen in particular to be $e^{in\theta} J_n(\gamma_{nm} r)$ with $J_n(\gamma_{nm} R) = 0$. It is then immediate to verify that Gauss law simply implies $p^{(0)}_{nm} \approx 0$.

We can also introduce creation-annihilation operators as in (2.17,2.18), where now $j = \alpha, \beta$, and

$$\Omega^{(\alpha)}_{nm} = \sqrt{\alpha^2_{nm} + m^2_H e^2} \;\; , \;\; \Omega^{(\beta)}_{nm} = \sqrt{\beta^2_{nm} + m^2_H e^2} \;\; , \;\; \Omega^{(h)}_{n} = m_H e \, . \qquad (3.22)$$

Therefore, the Hamiltonian (3.5) acting on the physical states becomes:

$$H = \Omega^{(\alpha)}_{nm} a^{(\alpha)\dagger}_{nm} a^{(\alpha)}_{nm} + \Omega^{(\beta)}_{nm} a^{(\beta)\dagger}_{nm} a^{(\beta)}_{nm} + \Omega^{(h)}_{n} a^{(h)\dagger}_{n} a^{(h)}_{n} \, , \qquad (3.23)$$

Notice that the Hamiltonian (3.23) has been explicitly derived from (3.5) for $\lambda = 0$. In this case, $\Omega_{nm}^{(\alpha)} = \Omega_{nm}^{(\beta)} = \Omega_{nm}$ just as in the massive vector meson case for $\lambda = 0$. For $\lambda > 0$, (3.5) assumes a form which is almost identical to (3.23), the only difference being that the harmonic modes are no longer solutions of (3.13) and hence do not appear in the Hamiltonian.

4 Conclusions

Let us now compare the vector meson model with the Higgs model in the London limit. From (2.15) and (3.21) it is clear that there is a one-to-one correspondence between the modes of the fields for these two theories (if we take into account the Gauss law $p_{nm}^{(0)} \approx 0$ which kills one set of modes for the Higgs theory). But what about the Hamiltonians?

It is known [1] that on an infinite plane these two models are indeed equivalent, both describing a massive electromagnetic potential A_μ. We see this also in our approach, by noting that the only BC's that are suitable to a plane geometry require that all the fields vanish at infinity and hence force both λ and μ to be zero. In the latter case, the Hamiltonian for the massive vector meson model (2.20) and the one for the Higgs model (3.23) are exactly the same, once we identify m_A with m_H. Therefore, both on an infinite plane and on a disc with BC's $\lambda = \mu = 0$, these two models coincide.

This is not the case if we confine the theory on a disc and impose BC's with $\lambda > 0$ (but still $\mu = 0$). The two Hamiltonians are then different: while for the Higgs model, (3.23) is diagonal, in the massive vector meson model it has the form (2.21), in which every mode of the electric field is coupled to infinitely many others. Thus the massive vector meson model and the Higgs model are equivalent on a disc only if we choose boundary conditions for the fields characterized by the value zero for the parameter(s) that appear in (2.10) and (3.11,3.12).

We would like to end this paper by briefly comparing the models under consideration to yet another model describing a massive vector meson, namely the Maxwell-Chern-Simons (MCS) theory. The action for this model reads:

$$S = \int_{D\times\mathbf{R}^1} d^3x \left\{ -\frac{1}{4e^2} F_{\mu\nu}F^{\mu\nu} + \frac{k}{4\pi}\epsilon^{\mu\nu\rho}A_\mu\partial_\nu A_\rho \right\} . \tag{4.1}$$

This Lagrangian has been studied in detail in [3], where it has been shown that the fields have to satisfy BC's characterized by a nonnegative parameter λ, exactly like in (2.10) or (3.11). As in section 2, one can show that the Hamiltonian of this system is diagonalized by the modes of the operator $*d*d$ only if $\lambda = 0$ and that, as soon as λ deviates from this value, the Hamiltonian couples an infinite number of such modes. In fact, the Hamiltonian for the MCS theory derived in [3] and the one for the massive vector meson model, (2.21), do coincide if we make the identifiction $m_A = \frac{ek}{2\pi}$. In particular, both these Hamiltonians concide with the Higgs Hamiltonian (3.23) when λ is chosen to be zero. In this latter case, the MCS theory admits an additional set of observables ("edge" observables) which commute with the Hamiltonian and are completely localized at the boundary of the spatial manifold D.

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